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by making CR and DX sufficiently small; or that the sect ST in passing from coincidence with CD to coincidence with RX is a continuous variable and so passes through all intermediate values.

Proposition VII. *The hypothesis of acute angle, if even in a single case it is true, always in every case it alone is true.*

This is very easily shown. For if the hypothesis of acute angle should permit any case of either other hypothesis, either of right angle, or of obtuse angle, now (from the two preceding propositions) no place would be left for the hypothesis of acute angle; which is absurd.

Therefore the hypothesis of acute angle, if even in a single case it is true, always in every case it alone is true.

Quod erat demonstrandum.

THE FIRST DIFFERENTIAL CO-EFFICIENT OF A CIRCLE.

By Professor JOHN N. LYLE, Ph. D., Professor of Natural Science, Westminster College, Fulton, Missouri.

The equation of the circle AQD, Fig. 1. referred to the Cartesian rectangular co-ordinate axes AY, AX is $y^2=2Rx-x^2$ .... (1).

Let a point move from A in the path of the curve $AQDK$. Further, let the motion be such that wherever possible dx, the rate of variation of the abscissa x, may be uniform.

y is a function of x, for it takes a new value corresponding to each new value assumed by the variable x.

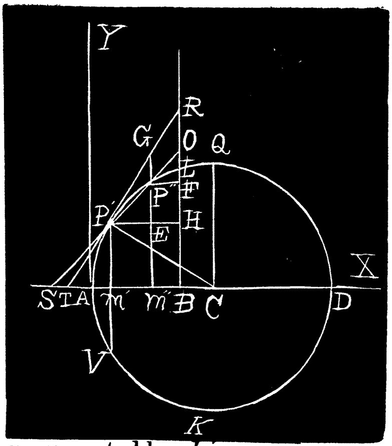
When the moving point has arrived at P' the variable x has reached the individual value x' or Am'; and the function y, the corresponding individual value y' or $P'm'$.

We then have $y'^2=2Rx'-x'^2$ ....(2).

Let dx' or $m'B$ stand for the increment to the variable x in the unit of time after reaching the value x' or Am'.

Since x increases uniformly, dx' represents its rate of increase.

Draw BR perpendicular to the axis of abscissas, intersecting the tangent line $TP'R$ in the point R. Draw $P'H$ perpendicular to BR. The line $P'H$ is equal to the constant line $m'B$, and is, consequently, represented by dx'.



Divide dx or $P'H$ into n equal parts. Then $\frac{dx}{n}$ or $\Delta x = \frac{P'H}{n}$ or $P'E$.

Erect the perpendicular EP'' to the line $P'H$ intersecting the curve in the point $P.''$ whose co-ordinates are x'', y''.

Through the points P' and P'' draw the secant line $SP'P''O$.

When the moving point has reached P'', the function y has reached the new value y'', or $y' + \Delta y'$, corresponding to the new value x'', or $x' + \frac{dx'}{n}$, assumed by the variable x.

The equation of the curve corresponding to the point P'' is $y''^2 = 2Rx'' - x''^2$ or $(y' + \Delta y')^2 = 2R(x' + \Delta x') - (x' + \Delta x')^2$, (3), in which $\Delta x'$ and $\Delta y'$ are the increments of the abscissa x and the ordinate y in one nth of the unit of time after the moving point reached P'.

Subtract (2) from (3).

Then $2y'\Delta y' + \Delta y'^2 = 2R\Delta x' - (2x'\Delta x' + \Delta x'^2)$, (4), and $2y'\Delta y' = 2(R - x')\Delta x' - \Delta x'^2 - \Delta y'^2$ and $\Delta y' = \frac{R - x'}{y'}\Delta x' - \frac{\Delta x'^2 + \Delta y'^2}{2y'}$, ...(5).

Multiply both members by n. Then $n \times \Delta y' = \frac{R - x'}{y'} dx' - \frac{n(\Delta x'^2 + \Delta y'^2)}{2y'}$..(6).

As n increases in value and $\Delta x'$ correspondingly diminishes, the product $n \times \Delta x'$, or dx, remains constant.

As n increases and $\Delta y'$ correspondingly diminishes, the product $n \times \Delta y'$ is seen from inspecting the second member of the (6) to be a variable that approaches the constant $\frac{R - x'}{y'} dx$ as its limit.

The value of the variable product $n \times \Delta y'$ consists of two parts, one of which is constant; and the other of which $\frac{n(\Delta x'^2 + \Delta y'^2)}{2y'}$ is a variable that decreases indefinitely without limit.

The part $\frac{R - x'}{y'} dx'$ is due to the tendency of y to vary when it reached the value y', that is, to the rate of variation of y when it equals y'.

$\frac{\Delta x'^2 + \Delta y'^2}{2y'}$ is due to the increment to the tendency of y to vary during one nth of the unit of time after reaching the value y'. This multiplied by n is $\frac{n(\Delta x'^2 + \Delta y'^2)}{2y'}$.

Let $i = \frac{n \times (\Delta x'^2 + \Delta y'^2)}{2y'}$.....(7).

Add (7) to (6) and we shall have $n\times \Delta y'+i=\frac{R-x'}{y'}\,dx'\ldots.(8)$.

The second member of equation (8) represents the rate of variation of y corresponding to the value y'.

For $n\times \Delta y'+i$ write dy'.

Then $dy'=\frac{R-x'}{y'}\,dx'\ \ldots.(9)$.

Hence, dy' stands for the rate of variation of the variable, y when it reaches the value y'.

dy' and dx' are called differentials of y and x corresponding to y' and x'.

The ratio between dy' and dx' is called the first differential co-efficient for the point P'.

Dropping the primes we have as a general expression for the first differential co-efficient $\frac{dy}{dx}=\frac{R-x}{y}\ \ldots.(10)$.

Remark 1. dx, or $n\times \Delta x'=P'H$ or $n\times P'E$, and $n\times \Delta y'=n\times P''E=HO$.

Remark 2. As n increases without limit, P'' approaches P', $P'E$ and $P''E$ diminishes without limit. $n\times P'E$ remains constant, and $n\times P'E$ or HO is a variable approaching the constant HR.

HO is the geometrical equivalent of $n\times\Delta y'$, and HR its limit is the geometrical equivalent of dy' the symbol representing the differential or rate of variation of the variable y corresponding to the value $P'm'$ or y'.

Remark 3. The limit of the increasing variable HO is obtained by adding these to the difference OR between HO and its limit, the constant HR.

The limit of the increasing variable $n\times \Delta y'$ is obtained by adding these to the difference i between $n\times \Delta y'$ and its limit the constant dy'.

Equation (8) is obtained from (7) and (6) in accordance with the Euclidian axiom—"If equals be added to equals, the wholes are equal."

Remark 4. Equation (6) is $n\times \Delta y'=$

$$\frac{R-x'}{y'}dx'-\frac{n(\Delta x'^2+\Delta y'^2)}{2y'}.$$

The geometrical equivalent of $n\times \Delta y'$ is HO.

Join $P'C$. Since $x'=Am'$, $R-x'=Cm'$. Hence the geometrical equivalent of $\frac{R-x'}{y'}dx'$ is $\frac{Cm'}{P'm'}\times PH$.

Remembering that the triangles $Cm'P'$ and RHP' are similar, the angle $m'C'P'$ being equal to HRP', we have $HR:P'H::Cm':P'm'$.

$$\text{Or } HR = \frac{Cm'}{P'm} \times P'H.$$

Hence HR, also, is the geometrical equivalent of $\frac{R-x'}{y'}\,dx'$.

Produce $P'm'$ to V and join Vp''.

Then $2y' = P'V$ and $\Delta x'^2 + \Delta y'^2 = P'P''^2$ and $n \times P'P'' = P'O$.

Hence $\frac{n \times (\Delta x'^2 + \Delta y'^2)}{2y'} = \frac{P'P''}{P'V} \times P'O$.

The angle $VP''P'$ is equal to $m'CP'$.

Hence, the triangle $P'OR$ and $VP'P''$ are similar and consequently $P'V : P'P'' :: P'O : OR$.

That is, $OR = \frac{P'P''}{P'V} \times P'O$.

It is evident, therefore, that the difference OR between the variable HO and its limit HR is equal to $\frac{P'P''}{P'V} \times P'O$ the difference between $n \times \Delta y'$ or $n \times P''E$ and its limit dy' or $\frac{Cm'}{P'm'} \times P'H$.

The line OR is the geometrical equivalent of i in equations (7) and (8).

Remark 5. The equality between the quantities i and $\frac{n(\Delta x'^2 + \Delta y'^2}{2y'}$, or between OR and $\frac{P'P''}{P'V} \times P'O$ explains what has been called the compensation of errors in the Leibnitzian method.

Bledsoe in his Philosophy of Mathematics, pages 155 and 156, shows that Bishop Berkeley deserves the credit of discovering and first stating this principle of the compensation of errors.

In Gillespie's translation of Compte's Philosophy of Mathematics, page 100, Compte says—"After various attempts more or less imperfect, a distinguished geometer, Carnot, presented at last the true direct logical explanation of the method of Leibnitz, by showing it to be founded on the principle of the necessary compensation of errors, this being, in fact, the precise and luminous manifestation of what Leibnitz had vaguely and confusedly preceived."

The demand of the Leibnitzian School that the line OR is so diminutive that the lines OH and RH are equal in length discredits Euclid's second axiom and stands as an obstacle in the way of correct theory of a the Calculus.

Remark 6. As n receives the successive finite values 10, 100, 1000, &c., P'' approaches P' and O approaches R.

As P'' approaches P' in accordance with the hypothesis in hand, the lines $P'm'$, $m'C$, CP', $P'H$, HR and $P'R$ remain constant, whilst $P'E$, $P''E$, $P'P''$ and OR simultaneously diminish without limit.

The angle $VP''P'$ is constant in value and equal to $P'RO$, whilst $P'VP''$ diminishes without limit.

The two following things may be stated respecting the series 10, 100, 1000, &c.:

1. Every term admitted into the series is a finite number.
2. No last term is attributed to the series.

The line $P'H$ is a constant. Since every value of n is assumed to be finite, every value of $P'E$, which is one nth of $P'H$, must be finite.

Whilst $P'E$ decreases in value without limit as n increases without limit, it is, nevertheless, *indestructible.*

Since by hypothesis it is a part of the constant, finite line $P'H$.

Furthermore, each of the lines $P'E$, and $P''E$, $P'P''$ and OR has *two* ends, and is, therefore, *finite.*

The hypothesis under which we are working prevents P'' from coinciding with P' and hence prevents the annihilation of $P'E$ and $P''E$.

The increments $P'E$ and $P''E$ are the data from which we logically infer the rate of variation of y for the point P. As we do not obtain logical conclusions by destroying the premises from which we reason it is not required of us to annihilate the indestructible lines $P'E$, $P''E$, or convert the secant $SP'P''O$ into the tangent $TP'R$.

We may draw a straight line through the point P'. If this line cuts the circle in two points, it is a secant; but if it touches the circle in one point only, it is a tangent.

We may also freely admit that a straight line through the point P' may be revolved about that point under a hypothesis that will permit it to make a complete revolution. But when it touches the circumference it does not cut; and when it cuts the circumference it is not a tangent thereto.

The revolving line must, in fact, cease to be a secant before it can be a tangent to the circle. Hence, the absurdity of assuming that a secant ever becomes a tangent.

According to Euclid's Proposition II. Book III.—"If any two points are taken in the circumference of a circle, the chord which joins them falls within the circle". If the straight line joining the points P' and P'' be extended both ways it will be a secant line, the portion of it within the circle being called a chord. Euclid's theorem holds however near P'' may approach P'. The facts of the Calculus rightly apprehended and correctly explained are found to be in harmony with the data of Euclid's Elements.

Remark 7. The tangent line to the circle at the point A is perpendicular to the axis AX, and $n\times\Delta y$ corresponding to that point is not a variable product having a constant quantity as a limit, but one that increases indefinitely without limit as n increases without limit and Δy correspondingly diminishes.

Is there a differential coefficient for the point A? Let us examine and report. For the point A the product $n\times\Delta y$ is not a constant line and is not a variable having a constant line as a limit. If the rate of variation of y wherever possible is either a constant equal to $n\times\Delta y$ or a constant and the limit of $n\times\Delta y$, then for the point A there is no rate of y. For that point $n\times\Delta y$ is a variable that increases indefinitely without limit. A differential coefficient is a ratio between tworates of variation. If either rate is *absent* for any point of the curve, there is no ratio and hence, no differential coefficient for that point.

The general expression for the differential coefficient of a circle is $\frac{dy}{dx}=\frac{R-x}{y}$.

For the point A this expression reduces to the form $\frac{dy}{dx}=\frac{R}{0}$.

This is an absurd collocation of analytical symbols inasmuch as a quotient is represented as possible in the absence of a divisor. This absurd expression $\frac{R}{0}$ occurs at the point A where the line that touches the curve is perpendicular to the axis of abscissas and may be taken to indicate that fact but not because $\frac{R}{0}$ is an expression for the tangent of 90°

The line AY that touches the quadrant AQ at A is not terminated by the line drawn through the centre and the point Q.

But does not the line CQ intersect AY "at infinity"? Euclid and Lobatschewesky agree in affirming that there is no intersection at any point whatever.

But is not the tangent of 90° ∞ ?

If the tangent of an arc is a line touching one extremity and terminated by a line drawn through the centre and the other extremity, it is evident that 90° has no tangent under this definition. Furthermore, a straight line having two ends is *finite* and cannot appropriately be represented by the hieroglyphic ∞.

Remark 8. Is there a differential co-efficient at the point Q?

The tangent line to the circle at the point Q is parallel to the axis AX

and $n \times \Delta y$ corresponding to that point is not a variable product having a constant quantity as a limit, but one that decreases indefinitely without limit as n increases without limit and Δy correspondingly diminishes. $n \times \Delta y$ for the point Q is not a constant quantity and is not a variable having a constant quantity as a limit. If either of these things be essential to a differential of y and hence to a differential co-efficient, then there can be no differential of y and no differential co-efficient corresponding to the point Q.

But does not the perpendicular to CQ at the point Q make a zero angle with the axis of abscissas? According to both Euclid and Lobatschewsky the two lines perpendicular to the radius CQ do not meet. If lines that make an angle with each other always meet, the perpendiculars to CQ do not make an angle.

What is a zero angle? Is it the ghost of a departed quantity that the Bishop of Cloyne tells about?

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ARITHMETIC.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

25. Proposed by L. B. HAYWARD, Superintendent of Schools, Bingham, Ohio.

A company engaged an agent to do business for one month at a salary of $25, giving him goods amounting to $57.54 and $32.17 in cash to start with. The agent bought during the month, goods amounting to $59.91. At the end of the month the goods on hand amounted to $31.67, and the amount of sales for the month was $102.97; what was the balance of account?

Solution by W. F. BRADBURY, A. M., Head-Master, Cambridge Latin School, Cambridge, Massachusetts.

Cash received	$32.17
Cash for sales	$102.97
Cash total	$135.14
Cash spent	$59.91
Cash on hand	$75.23
Salary of agent	$25.00
Goods to be returned and cash	$50.23

It is assumed that the agent paid for the goods he bought and re-